# CEGAR-Based Approach for Solving Combinatorial Optimization Modulo Quantified Linear Arithmetics Problems

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## Optimization Modulo Quantified Linear Arithmetics Problems — OPT+qLP

minimize $f_{\text{obj}}(x)$ | minimizing a Boolean objective function $f_{\text{obj}} : \mathbb{B}^n \to \mathbb{R}$

such that:

$$\bigwedge_c c(x)$$

**SAT problem**: SAT / ASP solvers, Boolean Optimization

$$\wedge \bigwedge_d d(x,y)$$

**SAT problem modulo linear constraints**: SMT solvers (*QF_LRA*), *Clingo[lpx]* **[1]**

$$\wedge\ \forall z \in \mathbb{R}^p, \bigwedge_e e(x,z) \implies \bigwedge_h h(x,z)$$

**SAT problem modulo quantified linear constraints**: *Restricted to one level of quantifiers*

SMT solvers and *Clingo[lpx]* with quantifier elimination

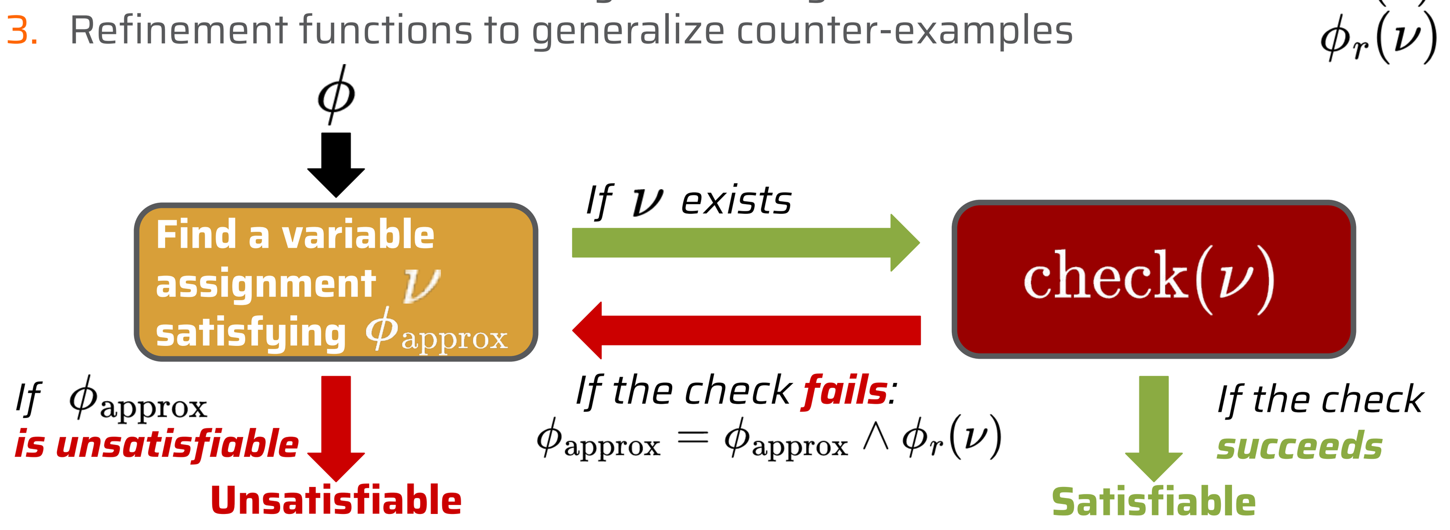
with $x \in \mathbb{B}^n, y \in \mathbb{R}^m$ $c : \bigvee_i x_i \vee \bigvee_j \neg x_j$

$d, e, h : \bigvee_i x_i \vee \bigvee_j \neg x_j \vee \sum_k \lambda_k \times y_k \leq 0$

## CEGAR [2]: *Counter-Example Guided Abstraction Refinement*

**Rely on:**

1. An over-approximation of the OPT+qLP problem — $\phi \implies \phi_{\text{approx}}$
2. Methods to check the validity of an assignment — $\text{check}(\nu)$
3. Refinement functions to generalize counter-examples — $\phi_r(\nu)$




- Similar to offline SMT approaches
- Already used to solve SMT problems **[3]**

**Advantages:**

- Solver independent
- Efficient for combinatorial problems

## Contribution: A CEGAR for Solving OPT+qLP problems

### 1. Over-Approximation

- Replace each linear constraint by an unique Boolean variable
- Replace $\implies$ by a $\wedge$ in the universally quantified linear constraint

### 2. Checking quantified linear constraints

**Definition**

$\mathcal{C}_i$: set of LP constraints that must hold for an assignment $x$ of $\phi_{\text{approx}}$ to satisfy $\bigwedge_i i(x,y)$

**Existential quantifier**

If $\mathcal{C}_d$ is satisfiable **accept**, else **reject**

**Universal quantifier**

1. Check if $\mathcal{C}_e$ is satisfiable
   a. if **no**, **accept**
   b. if **yes**, continue
2. For each $h \in \mathcal{C}_h$:
   a. $h^*$: maximum of $h$ under $\mathcal{C}_e$
   b. if **not** $h^* \leq 0$, **reject**
3. **accept**

***Note:*** *Linear optimization problems are solved by dedicated LP solvers*

### 3. Counter-example generalization

**Rely on 2 monotone properties:**

1. All supersets of an unsatisfiable set of linear constraints is unsatisfiable

**Unsatisfiable core:** smallest unsatisfiable subset of linear constraints

2. Adding constraints to a linear optimization problem cannot increase its maximum value

**Optimal core of** $(h, \mathcal{C})$: biggest superset of $\mathcal{C}$ having the same maximum value as $(h, \mathcal{C})$

**Example:** $\{\alpha,\beta\}$ and $\{\alpha,\gamma\}$ are optimal cores of $\{\alpha\}$
All their subsets are such that $\max y = \infty$

**Problem:** *computing **optimal** and **unsatisfiable cores** can be computationally expensive*

**Proposition: *linear constraint partitioning***
Partition the set of linear constraints into independent subsets, *i.e. no shared variables between subsets*

**EXAMPLE**

**OPT+qLP problem** $\phi$

$(a \vee b \vee c)$

$\wedge\ \forall x, y \in \mathbb{R}, \left( \begin{array}{l} (y \geq 1 \vee \neg a) \\ \wedge\ (x + y \leq 1 \vee \neg b) \\ \wedge\ (-x + y \leq 0 \vee \neg c) \end{array} \right) \implies y \leq 0.6$

with $a, b, c \in \mathbb{B}$

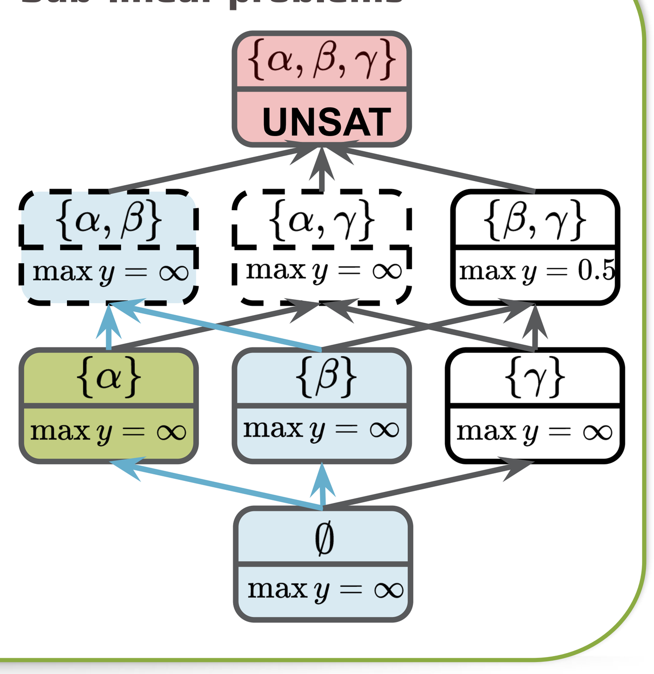
**Its over-approximation** $\phi_{\text{approx}}$

$(a \vee b \vee c)$

$\wedge\ (\alpha \vee \neg x_1) \wedge (\beta \vee \neg x_2) \wedge (\gamma \vee \neg x_3) \wedge \delta$

with $\alpha, \beta, \gamma, \delta \in \mathbb{B}$

**Sub-linear problems**




## Implementation and Benchmark

### Our implementation: *MerrinASP*

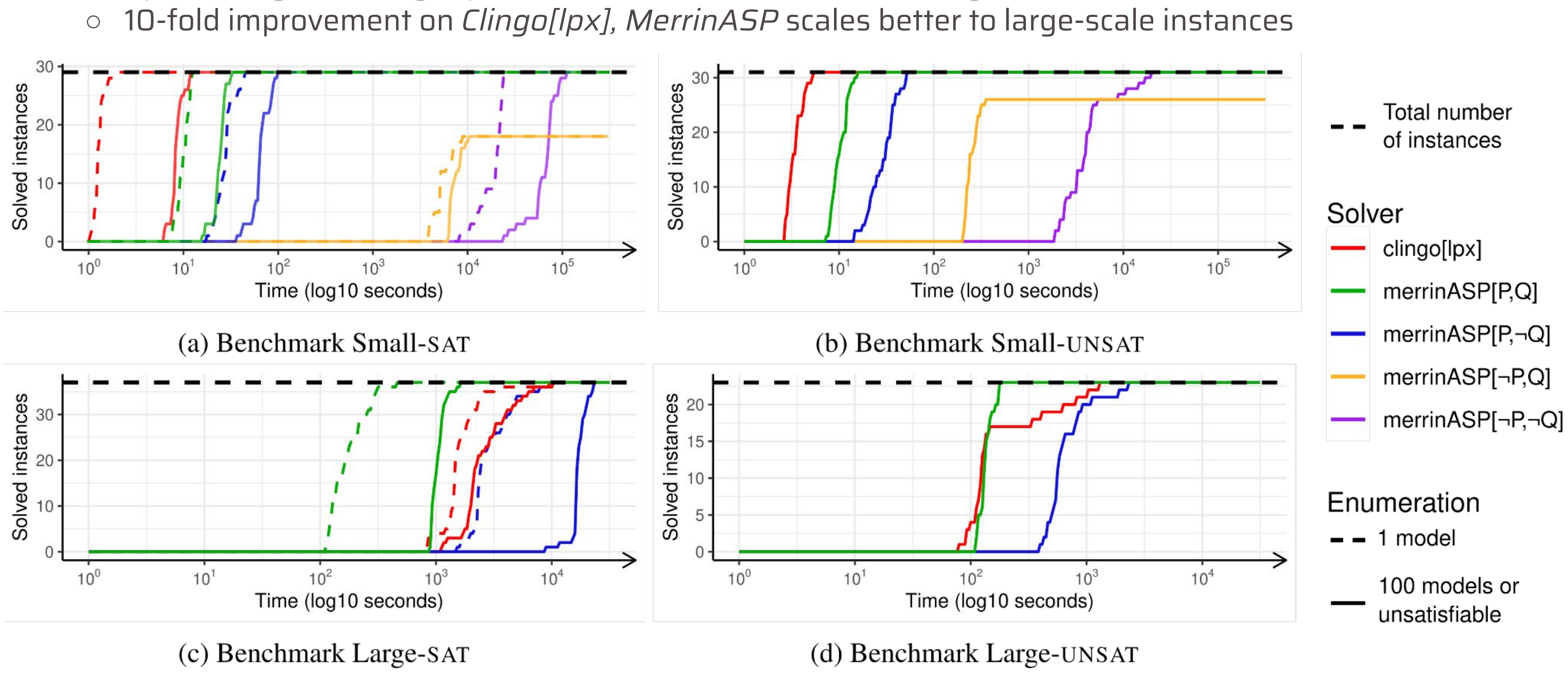
Extend ASP solver *clingo* **[4]** with:

- one-level of quantified linear constraints ;
- linear constraints partitioning.

Support different LP solvers (*e.g.* CPLEX, GUROBI, GLPK, etc).

### Benchmarks

- 2 benchmarks, each composed of 60 OPT+qLP problems inspired by System Biology **[5]**
- Comparison against *Clingo[lpx]* **[1]** and *MerrinASP* under 4 configurations
  - 10-fold improvement on *Clingo[lpx], MerrinASP* scales better to large-scale instances




(a) Benchmark Small-SAT (b) Benchmark Small-UNSAT (c) Benchmark Large-SAT (d) Benchmark Large-UNSAT

## Conclusion

- CEGAR-based approach to solve OPT+qLP problems
- Validate on a benchmark inspired by Bioinformatics
- *MerrinASP*: implementation available on *GitHub*

### Future Work

- Study the impact of line ar solver and its interface on performance
- Compare generate constraints with constraints generated by the LRA theory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***MerrinASP repository*** ***Proceeding***

**References**


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